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Abstract: A *good* sample is a point set such that any ball of radius ϵ contains a constant number of points. The Delaunay triangulation of a good sample is proved to have linear size, unfortunately this is not enough to ensure a good time complexity of the randomized incremental construction of the Delaunay triangulation. In this paper we prove that a random Bernoulli sample of a good sample has a triangulation of linear size. This result allows to prove that the randomized incremental construction needs an expected linear size and an expected $O(n \log n)$ time.

Key-words: Probabilistic analysis – Worst-case analysis – Randomized incremental constructions

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La triangulation de Delaunay d'un échantillon aléatoire d'un bon échantillon a une taille linéaire

Résumé : Un *bon* échantillon est un ensemble de points tel que toute boule de rayon ϵ contienne un nombre constant de points. Il est démontré que la triangulation de Delaunay d'un bon échantillon a une taille linéaire, malheureusement cela ne suffit pas à assurer une bonne complexité à la construction incrémentale randomisée de la triangulation de Delaunay. Dans ce rapport, nous démontrons que la triangulation d'un échantillon aléatoire de Bernoulli d'un bon échantillon a une taille linéaire. Nous en déduisons que la construction incrémentale randomisée peut être faite en temps $O(n \log n)$ et espace $O(n)$.

Mots-clés : Analyse probabiliste – Analyse dans le cas le pire – construction incrémentale randomisée

1 Introduction

A *good* sample of some domain $D \subset \mathbb{R}^d$ is a point set of size n in \mathbb{R}^d such that any ball centered in D of radius ϵ contains a constant number of points. When we enforce a hypothesis of good distribution of the points in space, a volume counting argument ensures that the local complexity of the Delaunay triangulation around a vertex is bounded by a constant (dependent on ϵ and d but not on the number of points). Unfortunately, to be able to control the complexity of the usual randomized incremental algorithms [1–4], it is not enough to control the final complexity of the triangulation, but also the complexity of the triangulation of a random subset.

One would expect that a random sample of size k of a good sample is also a good sample with high probability. Actually this is not quite true, it may happen with reasonable probability that balls of volume $O\left(\frac{1}{k}\right)$ may contain $\log n$ points or that a ball of volume $\Omega\left(\frac{\log k}{k}\right)$ does not contain any point. Thus this approach can transfer the complexity of a good sample to the one of a random sample of a good sample but losing log factors.

In this paper, we study directly the Delaunay triangulation of a random sample of a good sample and deduce results about the complexity of randomized incremental constructions.

2 Results, Definitions, and Notations

We define several sampling notions, our point set is in \mathbb{T}^d the flat torus of dimension d to avoid boundary conditions ($\mathbb{T} = \mathbb{R}/\mathbb{Z}$):

Definition 1. A set \mathcal{X} of n points in \mathbb{T}^d is an (ϵ, κ) -sample if any ball of radius ϵ contains at least one point and at most κ points.

Definition 2. A subset \mathcal{Y} of set \mathcal{X} is a Bernoulli sample of \mathcal{X} of parameter α if each point of \mathcal{X} belongs to \mathcal{Y} with probability α independently.

Definition 3. A subset \mathcal{Y} of set \mathcal{X} is a uniform sample of \mathcal{X} of size k if \mathcal{Y} is any possible subset of \mathcal{X} of size k with equal probability.

This short note proves the two following theorems:

Theorem 4. Given an (ϵ, κ) -sample \mathcal{X} in \mathbb{T}^d , a Bernoulli sample \mathcal{Y} of probability α of \mathcal{X} , and a point p , the expected number of faces of dimension i of the Delaunay triangulation of $\mathcal{Y} \cup \{p\}$ that contain p is at most $72^{di+O(d+i(\log i\kappa))}$. In particular this number is a constant with respect to n and α .

Theorem 5. Given an (ϵ, κ) -sample \mathcal{X} in \mathbb{T}^d , the randomized incremental construction of the Delaunay triangulation needs $O(n \log n)$ expected time and $O(n)$ expected space.

An analogous of Theorem 4 for uniform samples instead of Bernoulli samples is probably true but more difficult to prove. Since Theorem 4 suffices to deduce Theorem 5, we leave the result for uniform sample as a conjecture.

We denote by $\Sigma(p, r)$ and $B(p, r)$ the sphere and the ball of center p and radius r respectively. $\#(Z)$ denotes the cardinality of the set Z .

The volume of the unit ball of dimension d is denoted V_d , the area of its boundary is denoted S_{d-1} , the formulas are $V_d = \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}$ and $S_d = 2\pi V_{d-1}$.

The diagram illustrates Lemma 7. It features two main regions: a light blue circle labeled $\Sigma(x, 3)$ and a red circle labeled $\Sigma(z, r)$. Inside the blue circle, there are three smaller circles: a green one containing points x , y , and w ; a pink one containing point z ; and a white one containing point p . Distances between these points are indicated by arrows: $|x-y| \leq 1$, $|y-w| \leq 1$, $|w-z| \leq 3$, $|z-p| \leq 1$, and $|p-x| \leq 3$. Outside the blue circle, there are additional circles and a distance of $\frac{\sqrt{35}}{3}$ from a point to the boundary.

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where the last line uses the lower bound of Lemma 8. \square

Proof of Theorem 4. If the farthest Delaunay neighbor of p is at distance less than $6r$, and $k = \sharp(\mathcal{Y} \cap B(p, 6r))$ then k^i is a trivial bound on the number of Delaunay i -faces incident to p . Thus we will compute the following:

$$\begin{aligned} & \mathbb{E} [\sharp(\text{Delaunay } i\text{-faces incident to } p)] \\ & \leq \mathbb{E} [\sharp(\mathcal{Y} \cap B(p, 6\epsilon))^i] + \sum_{j=1}^{\infty} \mathbb{P} [F(p, 6 \cdot 2^{j-1}\epsilon)] \mathbb{E} [\sharp(\mathcal{Y} \cap B(p, 6 \cdot 2^j\epsilon))^i \mid F(p, 6 \cdot 2^{j-1}\epsilon)] \end{aligned}$$

On the one hand, we remark that

$$\mathbb{E} [\sharp(\mathcal{Y} \cap B(p, 2r)) \mid F(p, r)] \leq \mathbb{E} [\sharp(\mathcal{Y} \cap B(p, 2r))],$$

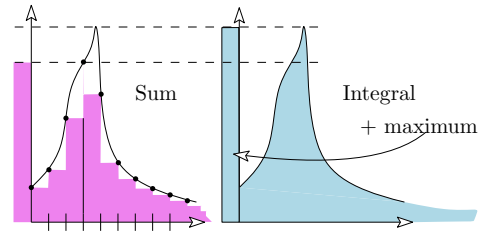
since the knowledge that the farthest neighbor is far implies that there are some points of \mathcal{X} that do not belong to the sample \mathcal{Y} and biases the probability in the above direction. On the other hand Lemma 2.2 in [5] allows to deduce for any domain D :

$$\mathbb{E} [\sharp(D \cap \mathcal{Y})^i] \leq i^i \mathbb{E} [\sharp(D \cap \mathcal{Y})]^i.$$

It gives:

$$\begin{aligned} & \mathbb{E} [\sharp(\text{Delaunay } i\text{-faces incident to } p)] \\ & \leq i^i \mathbb{E} [\sharp(\mathcal{Y} \cap B(p, 6\epsilon))^i] + \sum_{j=1}^{\infty} \mathbb{P} [F(p, 6 \cdot 2^{j-1}\epsilon)] i^i \mathbb{E} [\sharp(\mathcal{Y} \cap B(p, 6 \cdot 2^j\epsilon))^i] \\ & \leq i^i \alpha^i \sharp(\mathcal{X} \cap B(p, 6\epsilon))^i + \sum_{j=1}^{\infty} \mathbb{P} [F(p, 6 \cdot 2^{j-1}\epsilon)] i^i \alpha^i \sharp(\mathcal{X} \cap B(p, 6 \cdot 2^j\epsilon))^i \\ & \leq i^i \alpha^i \kappa^i 18^{di} + \sum_{j=0}^{\infty} A_d \exp(-\alpha (2^{j-2})^d) i^i \alpha^i \kappa^i (18 \cdot 2^j)^{di} \quad \text{using Lemmas 7 and 10} \\ & \leq i^i \alpha^i \kappa^i 18^{di} \left(1 + A_d \sum_{j=0}^{\infty} 2^{dij} \exp(-\alpha 2^{dj-2d}) \right). \end{aligned}$$

The function $j \rightsquigarrow 2^{dij} \exp(-\alpha 2^{dj-2d})$ is increasing between 0 and j_0 and decreasing between j_0 and ∞ , with $j_0 = 2 + \frac{1}{d} \log_2(\frac{i}{\alpha})$. Notice that the maximum of the function is $2^{dj_0} \exp(-\alpha 2^{dj_0-2d}) = 2^{2di+i \log_2(\frac{i}{\alpha})} \exp(-\alpha 2^{\log_2(\frac{i}{\alpha})}) \leq 4^{di} (\frac{i}{\alpha})^i$. A standard sum to integral comparison yields:



$$\begin{aligned}
& \mathbb{E} [\#(\text{Delaunay } i\text{-faces incident to } p)] \\
& \leq i^i \alpha^i 18^{di} \kappa^i \left(1 + A_d 2^{dij_0} \exp(-\alpha 2^{dj_0-2d}) + A_d \int_{j=0}^{\infty} 2^{dij} \exp(-\alpha 2^{dj-2d}) dj \right) \\
& \leq i^i \alpha^i 18^{di} \kappa^i \left(1 + A_d 4^{di} \left(\frac{i}{\alpha}\right)^i + A_d \int_{k=\alpha 2^{-2d}}^{\infty} \left(\frac{k 2^{2d}}{\alpha}\right)^i \exp(-k) \frac{dk}{k d \ln 2} \right) \quad \text{with } k = \alpha 2^{dj-2d} \\
& \leq i^i \kappa^i 72^{di} \left(1 + A_d i^i + A_d \int_{k=0}^{\infty} k^{i-1} \exp(-k) \frac{1}{d \ln 2} dk \right) \\
& \leq i^i \kappa^i 72^{di} \left(1 + A_d i^i + A_d \frac{(i-1)!}{d \ln 2} \right) \leq 3i^{2i} \kappa^i 72^{di} A_d.
\end{aligned}$$

□

4 From Bernoulli Sample to Uniform Sample of Size k

Proof of Theorem 5. Space complexity. Theorem 4 does not apply directly to the randomized construction of the Delaunay triangulation of \mathcal{X} . When the points are inserted in a random order, the k^{th} inserted point p_k is a random point in a uniform sample of size k of \mathcal{X} , the expected number of simplices of dimension i created by its insertion is $\mathbb{E} [\#(\text{Delaunay } i\text{-faces incident to } p_k \mid \#(\mathcal{Y}) = k)]$. To deduce the expected number of simplices created during the randomized incremental construction we have to sum on i and k .

The time complexity can be split in a location part and a construction part. The construction time is of the same order as the space complexity. The location time of an insertion in the usual randomized incremental construction of the Delaunay triangulation of a set \mathcal{X} of n points relies on a backward analysis argument. In the classical history graph, the argument is as follows: when locating the n^{th} point p_n , we trace the conflicts¹ within the history of the construction of the Delaunay triangulation; the insertion of p_k ($2 \leq k < n$) creates m conflicts with p_n if and only if $p_k p_n$ is an edge of the triangulation of $\{p_1, p_2, \dots, p_k, p_n\}$ incident to m simplices. Since p_k is a random point q in $\{p_1, p_2, \dots, p_k\}$ we get as expected location time:

$$\begin{aligned}
& \sum_{k=2}^{n-1} \mathbb{E} \left[\sum_{q \in \mathcal{Y}; \#(\mathcal{Y})=k} \mathbb{P}[p_k = q] \cdot \mathbb{1}_{[qp_n \text{ edge of } DT(\mathcal{Y} \cup \{p_n\})]} \cdot \#(\text{Delaunay } d\text{-simplices incident to edge } qp_n) \right] \\
& \leq \sum_{k=2}^{n-1} \frac{1}{k} \cdot d \cdot \mathbb{E} [\#(\text{Delaunay } d\text{-simplices incident to } p_n \text{ in } DT(\mathcal{Y} \cup \{p_n\}) \mid \#(\mathcal{Y}) = k)]
\end{aligned}$$

Bernoulli to uniform sampling. Thus, the number of simplices incident to a vertex needs to be controlled in the triangulation of a uniform sample of size k and not of a Bernoulli sample of probability α . Let f_k and g_α be the following random variables:

$$f_k = \#(\text{Delaunay } i\text{-faces of } DT(\mathcal{Y} \cup \{p\}) \text{ incident to } p),$$

where p is any point in the plane and \mathcal{Y} is a uniform random sample of \mathcal{X} of size k , and

$$g_\alpha = \#(\text{Delaunay } i\text{-faces of } DT(\mathcal{Y} \cup \{p\}) \text{ incident to } p)$$

¹ A point is in conflict with a simplex if it belongs to its circumscribing ball.

where p is any point in the plane and \mathcal{Y} is a Bernoulli sample of \mathcal{X} of parameter α . Then the classical equation is

$$\mathbb{E}[g_\alpha] = \sum_{k=0}^n \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \mathbb{E}[f_k] = \sum_{k=1}^n \binom{n}{k} \alpha^k (1-\alpha)^{n-k} \mathbb{E}[f_k]$$

since $\binom{n}{k} \alpha^k (1-\alpha)^{n-k}$ (the binomial distribution) is the probability that a Bernoulli sample of parameter α has size k . Theorem 4 gives $\mathbb{E}[g_\alpha] = O(1)$.

First, we can prove that the expected number of simplices constructed in a randomized incremental construction is $O(n)$:

$$\begin{aligned} O(1) = \int_0^1 \mathbb{E}[g_\alpha] d\alpha &= \sum_{k=1}^n \binom{n}{k} \mathbb{E}[f_k] \int_0^1 \alpha^k (1-\alpha)^{n-k} d\alpha \\ &= \sum_{k=1}^n \binom{n}{k} \mathbb{E}[f_k] \frac{1}{(n+1)\binom{n}{k}} = \frac{1}{n+1} \sum_{k=1}^n \mathbb{E}[f_k] \end{aligned}$$

Then, we can prove that the expected complexity of localizing the last point in the Delaunay triangulation of a uniform sample of size k is $O(\log n)$.

We can write

$$\begin{aligned} \int_0^1 \frac{\mathbb{E}[g_\alpha]}{\alpha} d\alpha &= \sum_{k=1}^n n \binom{n-1}{k-1} \frac{\mathbb{E}[f_k]}{k} \int_0^1 \alpha^{k-1} (1-\alpha)^{n-k} d\alpha \\ &= \sum_{k=1}^n n \binom{n-1}{k-1} \frac{\mathbb{E}[f_k]}{k} \frac{(n-k)!(k-1)!}{n!} = \sum_{k=1}^n \frac{\mathbb{E}[f_k]}{k}. \end{aligned}$$

To compute $\int_0^1 \frac{\mathbb{E}[g_\alpha]}{\alpha} d\alpha$ we can use $\mathbb{E}[g_\alpha] = O(1)$ (Theorem 4) except in the neighborhood of 0 where we use a separate, naive bound

$$\mathbb{E}[g_\alpha] \leq \mathbb{E}[\sharp(\mathcal{Y})^d] \leq d^d \mathbb{E}[\sharp(\mathcal{Y})]^d = d^d (\alpha n)^d$$

using again Lemma 2.2 from [5].

$$\begin{aligned} \int_0^1 \frac{\mathbb{E}[g_\alpha]}{\alpha} d\alpha &= \int_0^{\frac{1}{n}} \frac{\mathbb{E}[g_\alpha]}{\alpha} d\alpha + \int_{\frac{1}{n}}^1 \frac{\mathbb{E}[g_\alpha]}{\alpha} d\alpha \\ &\leq \int_0^{\frac{1}{n}} d^d \alpha^{d-1} n^d d\alpha + \int_{\frac{1}{n}}^1 \frac{O(1)}{\alpha} d\alpha \\ &= d^{d-1} + O(1) \cdot \log n = O(\log n). \end{aligned}$$

□

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